4. Find the $r^{th}$ moment about the origin for the RV with pdf $f(x) = kx^2 e^{-x}$, $x > 0$. Hence find first moments about the origin

Solution:

<u>To find 'k'</u>

By the definition, we've $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\Rightarrow k \int_0^{\infty} x^2 e^{-x}\,dx = 1.$$

$$\Rightarrow k \left\{ -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right\}_0^{\infty} = 1.$$

$$\Rightarrow k \{ 0 - (-2) \} = 1.$$

$$2k = 1$$

$$k = \frac{1}{2}.$$

$$\therefore f(x) = \frac{1}{2} x^2 e^{-x}, \ x > 0$$

$u = x^2$ $\quad$ $v = e^{-x}$

$(+)$

$u' = 2x$ $\quad$ $v_1 = \frac{e^{-x}}{-1}$

$(-)$

$u'' = 2$ $\quad$ $v_2 = \frac{e^{-x}}{(-1)^2}$

$(+)$

$u''' = 0$ $\quad$ $v_3 = \frac{e^{-x}}{-1}$

$r^{th}$ Moment about the Origin

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx = \frac{1}{2}\int_0^{\infty} x^r x^2 e^{-x}\,dx.$$

$$= \frac{1}{2}\int_0^{\infty} x^{r+2} e^{-x}\,dx.$$

$$\mu_r' = \frac{1}{2}\int_0^{\infty} e^{-x} x^{r+2}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} e^{-x} x^{(r+3)-1}\,dx.$$

$$\boxed{\mu_r' = \frac{1}{2}\Gamma(r+3).}$$

Since by the definition of Gamma function, we've

$$\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx.$$

also $\Gamma(n) = (n-1)!$

$\Gamma(n+1) = n!$

To find the moment about the origin

$$\mu_r' = \frac{1}{2}\Gamma(r+3) \quad \text{--- (1)}.$$

When $r=1$, (1) $\Rightarrow$ $\mu_1' = \frac{1}{2}\Gamma(4)$

$$= \frac{1}{2}(3)! = \frac{1}{2}(6).$$

$$\mu_1' = 3.$$

When $r=2$, (1) $\Rightarrow$ $\mu_2' = \frac{1}{2}\Gamma(5) = \frac{1}{2}(4)!$

$$= \frac{1}{2}(24).$$

$$\mu_2' = 12$$

When $r=3$, (1) $\Rightarrow$ $\mu_3' = \frac{1}{2}\Gamma(6) = \frac{1}{2}(5!).$

$$= \frac{1}{2}(120).$$